Phys 410
Fall 2015
Lecture #20 Summary
5 November, 2015

We then went through the steps of treating the classical Inductor-Capacitor ($L - C$) parallel circuit as a quantum problem. We chose for the generalized coordinate the flux in the inductor Φ, and employed a mechanical analog to identify the kinetic energy of the system as the energy stored in electric fields in the capacitor $U_C = \frac{1}{2}C\dot{\Phi}^2$, and the energy stored in magnetic fields in the inductor as the potential energy $U_L = \frac{1}{2}\frac{\Phi^2}{L}$. The Lagrangian is $\mathcal{L}(\Phi, \dot{\Phi}) = \frac{1}{2}C\dot{\Phi}^2 - \frac{1}{2}\frac{\Phi^2}{L}$. From this one can find the conjugate momentum as $p = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C\dot{\Phi} = Q$, where Q is the charge on the capacitor plate. Note that the generalized coordinate has the units of flux, while the conjugate momentum has the units of charge! Now we can derive the Hamiltonian and express it in terms of the generalized coordinate and its conjugate momentum as $\mathcal{H}(\Phi, \mathrm{Q}) = \frac{Q^2}{2C} + \frac{1}{2}\frac{\Phi^2}{L}$. Dirac's prescription is to observe that the coordinate and its conjugate momentum have a Poisson bracket of unity, hence the corresponding QM operators have this relationship: $\widehat{\Phi}\widehat{\mathrm{Q}} - \widehat{\mathrm{Q}}\widehat{\Phi} = i\hbar[\Phi, \mathrm{Q}]$, with $[\Phi, \mathrm{Q}] = 1$. The Hamiltonian can be re-written in terms of the creation and annihilation operators as $\widehat{\mathcal{H}} = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$, which is the Hamiltonian of a harmonic oscillator with $\omega = 1/\sqrt{LC}$. The eigenvalues of this operator are $\hbar\omega\left(n + \frac{1}{2}\right)$, where n is the number of excitations in the harmonic oscillator eigenfunction $|\psi_n\rangle$ and n runs from 0 to infinity.

We next considered the most general motion of systems of particles. We specifically consider rigid bodies, defined as multi-particle objects in which the distance between any two particles never changes as the object moves. As discussed before, this puts a huge constraint on the system, changing it from a 3N degree of freedom object to a 6 degree of freedom system. We reviewed the center of mass, center of mass momentum, and Newton's second law for the CM. We then considered the angular momentum of a rigid body and found that it decomposes cleanly into the angular momentum of the center of mass (relative to some chosen origin), and the angular momentum relative to the CM. For a rigid body, the only motion it can have relative to the CM is rotation. A similar observation was made about the kinetic energy of the rigid body.

Next we considered an arbitrary rigid object that is forced to rotate about a single fixed axis, which we take to be the z-axis. Because we want to work in an inertial reference frame, we choose the origin of the coordinate system to be somewhere on the axis of rotation. The angular velocity of the object can be written as $\vec{\omega} = \omega\hat{z}$. Naively we might expect that the angular momentum of the object to be $\vec{L} = I_z \vec{\omega}$, where $I_z = \sum_\alpha^N m_\alpha \rho_\alpha^2$ is the moment of inertia for rotation about that axis. This turns out to be true only in special cases of very symmetric objects, or when the axis of rotation is chosen along one of the 'principal axes', defined later. We did the full general calculation of $\vec{L}$ and found that $\vec{L} = L_x \hat{\imath} + L_y \hat{\jmath} + L_z \hat{k}$, where $L_x = -\sum_\alpha^N m_\alpha x_\alpha z_\alpha \,\omega$, $L_y = -\sum_\alpha^N m_\alpha y_\alpha z_\alpha \,\omega$, and $L_z = -\sum_\alpha^N m_\alpha (x_\alpha^2 + y_\alpha^2)\,\omega$. Thus in general the angular momentum vector $\vec{L}$ is not parallel to the axis of rotation $\hat{\omega}$. We perceive this as a "wobbling" motion because your efforts to bring such a rotating object to rest do not simply involve application of a torque to counteract the motion about the z-axis.

Next we considered an arbitrary rigid body rotating about an arbitrary axis (in general the axis of rotation of an object will change as it moves). Again we choose the origin to be on the axis of rotation (not necessarily at the center of mass!), allowing us to use the result that $\vec{v} = \vec{\omega} \times \vec{r}$, which only holds for this choice of origin. We calculated $\vec{L}$ by summing over all particles in the system (and using $\vec{v} = \vec{\omega} \times \vec{r}$) and found that the vector quantity could be broken down into components as $L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$, with $I_{xx} = \sum_\alpha^N m_\alpha (y_\alpha^2 + z_\alpha^2)$, $I_{xy} = -\sum_\alpha^N m_\alpha x_\alpha y_\alpha$, $I_{xz} = -\sum_\alpha^N m_\alpha x_\alpha z_\alpha$, and similar expressions for L_y and L_z. All of these results can be summarized in a simple matrix equation as $\vec{L} = \bar{\bar{I}}\vec{\omega}$, where $\vec{L} = \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix}$ is the angular momentum represented as a column vector, $\bar{\bar{I}} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$ is called the inertia tensor, and $\vec{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$ is the angular velocity vector. Note that the inertia tensor is symmetric about the diagonal: $I_{ij} = I_{ji}$. The inertia tensor is a property of the object and its mass distribution alone (<u>and</u> the choice of origin). $\vec{L} = \bar{\bar{I}}\vec{\omega}$ is a general expression relating the angular momentum vector to the axis of rotation.

We did the example of a cube of side a and mass M rotated about one edge. The inertia tensor can be calculated by converting the sums to integrals, for example: $I_{xx} = \sum_\alpha^N m_\alpha (y_\alpha^2 + z_\alpha^2) \xrightarrow{yields} \int_0^a dx \int_0^a dy \int_0^a dz \;\rho\,(y^2 + z^2)$, where $\rho = M/a^3$ is the density of the uniform cube (not to be confused with ρ_α above!). Here we assume that the corner of the cube (at the origin of the Cartesian coordinate system) will remain fixed during the rotation. The resulting inertia tensor for this case is $\bar{\bar{I}} = \frac{Ma^2}{12}\begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}$. This inertia tensor can be used for any rotation axis that passes through the corner of the cube at the origin. In particular, for rotation about the x-axis, $\vec{\omega} = (\omega, 0,0)$ and we find the angular momentum to be $\vec{L} = Ma^2\omega\left(\frac{2}{3}, -\frac{1}{4}, -\frac{1}{4}\right)$. It is clear in this case that $\vec{L}$ is not parallel to $\vec{\omega}$. This is due in part to the fact that the object is not symmetric with respect to the axis of rotation. On the other hand, if we choose the rotation axis to be along the body diagonal of the cube (through the corner where the origin is located) we have $\vec{\omega} = \frac{\omega}{\sqrt{3}}(1,1,1)$ and the resulting angular momentum vector is $\vec{L} = \bar{\bar{I}}\vec{\omega} = \frac{Ma^2}{6}\vec{\omega}$. For this choice of rotation axis the angular momentum vector IS parallel to the angular velocity direction. This gives us hope that there can be choices of rotation axes $\vec{\omega}$ such that the angular momentum vector is parallel to $\vec{\omega}$.